

# New pairs of matrices with convex generalized numerical ranges

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## Abstract

In this article, we are going to search for  $n \times n$  matrices  $A$  and  $B$  such that their generalized numerical range

$$W_A(B) = \{\operatorname{tr} AU^*BU : U^*U = UU^* = I\}$$

is convex. More specifically, we consider  $A = \hat{A} \oplus 0_{n-2}$  and  $B = \hat{B} \oplus 0_{n-2}$  where  $\hat{A}$  and  $\hat{B}$  are  $2 \times 2$ . If  $W_A(B) = W_{\hat{A}}(\hat{B})$  then it is a convex set.

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## 1 Introduction

Let  $M_n$  be the space of all  $n \times n$  matrices with standard basis  $\{E_{11}, E_{12}, \dots, E_{nn}\}$ , and  $U_n$  be the group of all  $n \times n$  unitary matrices.

For  $B \in M_n$ , the classical numerical range of  $B$  is the set

$$W(B) = \{x^*Bx : x \text{ is a unit vector}\}.$$

The classical numerical range is a compact set which contains all the eigenvalues of  $B$ , and it is a convex set by the famous Toeplitz-Hausdorff Theorem [4, 12]. See [5, Chapter 1] for a nice discussion.

Note that  $W(B) = \{\operatorname{tr} E_{11}X : X \in U(B)\}$  where  $U(B) = \{VBV^* : V \in U_n\}$  is the unitary orbit of  $B$ . This inspires the following generalization. Let  $C \in M_n$ , the set

$$W_C(B) = \{\operatorname{tr} CX : X \in U(B)\}$$

is called the  $C$ -numerical range of  $B$ . Therefore the classical numerical range of  $B$  is the  $E_{11}$ -numerical range of  $B$ . Note that the  $C$ -numerical range of  $B$  is the  $B$ -numerical range of  $C$ .

In 1975, Westwick [15] showed that if  $C$  is Hermitian then  $W_C(B)$  is convex. (See another proof by Poon [11].) Hence  $W_C(B)$  is convex if  $C$  is a normal matrix with collinear eigenvalues. Conjectured by Marcus [8] in 1975 and confirmed by Au-Yeung and Tsing [1] in 1983, if  $C$  is normal and  $W_C(B)$  is convex for all  $B$  then the eigenvalues of  $C$  must be collinear.

In 1984, Tsing [14] proved that if  $C$  is rank one then  $W_C(B)$  is convex for all  $B$ . A consequence is that  $W_C(B)$  is convex for any  $B, C \in M_2$ .

Problem 1. *Find more  $B, C$  with convex  $W_C(B)$ .*

Problem 2. *So far, for all  $B, C$  with convex  $W_C(B)$ , one of  $B$  and  $C$  must have collinear eigenvalues. Is it a general rule?*

In 1991, Li and Tsing [7] showed that if  $C = \lambda I + C_0$  where  $C_0$  is the block-shift form matrix then  $W_C(B)$  is always a circular disc centered at  $\lambda \mathbf{tr} B$ . Indeed, if  $W_C(C^*)$  is a circular disc centered at 0 then  $C$  must be a shift-block form matrix.

Problem 3. *Suppose  $W_C(C^*)$  is a circular disc. Does  $C = \lambda I + C_0$  where  $C_0$  is a block-shift from matrix?*

Although  $W_C(B)$  may fail to be convex, it is proved in 1981 by Tsing [13] that if  $C$  is normal then  $W_C(B)$  is star-shaped. Later in 1996, Cheung and Tsing [2] showed that  $W_C(B)$  is star-shaped for all  $C$  and  $B$ .

In this article, we will do Problem 1, i.e., to search other pairs of  $B, C \in M_n$  such that  $W_C(B)$  is convex. More specifically, we consider:

Problem 4. *Find  $A, B \in M_2$  such that  $W_{A \oplus 0_k}(B \oplus 0_k) = W_A(B)$ .*

If we can find such  $A, B$  then  $(A \oplus 0_k, B \oplus 0_k)$  is a “convex pair”.

In the end, we will answer Problem 2 and Problem 3 as well.

Let's have some more notations: For  $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in M_2$ , we write

$$B_0 = B - \left( \frac{1}{2} \mathbf{tr} B \right) I \quad \text{and} \quad B(\epsilon) = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} B \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} = \begin{pmatrix} b_{11} & \epsilon b_{12} \\ \epsilon b_{21} & \epsilon^2 b_{22} \end{pmatrix}$$

where  $0 \leq \epsilon \leq 1$ .

Recall that the numerical radius of a square matrix  $A$  is given by

$$r(A) = \max_{x \in W(A)} |x|.$$

## 2 Lemmas

In 1932, Murnaghan [9] proved the original Elliptical Range Theorem, which states that the classical numerical range of  $A \in M_2$  is an elliptical disc centered at  $\frac{1}{2}\text{tr } A$  and the two eigenvalues are the foci on the major axes. In 1994, Nakasato [10] generalized it to general  $W_C(A)$ . Let's state Nakasato's result as our first lemma.

**Lemma 1** (Elliptical Range Theorem) *Let  $A, B \in M_2$ . If  $A = \mu \begin{pmatrix} a & a_{12} \\ a_{21} & a \end{pmatrix}$  and  $B = \nu \begin{pmatrix} b & b_{12} \\ b_{21} & b \end{pmatrix}$  with  $a_{12} \geq a_{21} \geq 0$  and  $b_{12} \geq b_{21} \geq 0$  then*

$$W_A(B) = 2\mu\nu W \left( \begin{pmatrix} ab & a_{12}b_{12} \\ a_{21}b_{21} & b \end{pmatrix} \right) = 2ab + \mathcal{U}$$

*which is an elliptical disc centered at  $2ab$  and*

$$\mathcal{U} = W_A(B_0) = W_B(A_0) = W_{A_0}(B_0)$$

*is an elliptical disc centered at 0.*

Please also see another proof of Lemma 1 by Li [6].

**Lemma 2** [2] *Let  $A, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in M_2$ . We have  $W_A \left( \begin{pmatrix} b_{11} & \epsilon b_{12} \\ \epsilon b_{21} & b_{22} \end{pmatrix} \right) \subseteq W_A(B)$  where  $0 \leq \epsilon \leq 1$ .*

From now on, we always assume that  $n \geq 3$ .

**Lemma 3** *Let  $A, B \in M_2$  and  $U \in U_n$ . There exists  $\hat{A} \in U(A)$  and  $\hat{B} \in U(B)$  such that*

$$\text{tr} (A \oplus 0_{n-2}) U^* (B \oplus 0_{n-2}) U = \alpha \text{tr} \hat{A}(\epsilon) \hat{B} = \alpha \text{tr} \hat{A} \hat{B}(\epsilon)$$

*for some  $0 \leq \alpha, \epsilon \leq 1$ . If  $n = 3$ , then  $\alpha = 1$ .*

*Consequently, we have*

$$W_{A \oplus 0_{n-2}}(B \oplus 0_{n-2}) = \bigcup_{\hat{A} \in U(A), 0 \leq \alpha, \epsilon \leq 1} \alpha W_B(\hat{A}(\epsilon)) = \bigcup_{\hat{B} \in U(B), 0 \leq \alpha, \epsilon \leq 1} \alpha W_A(\hat{B}(\epsilon))$$

*when  $n \geq 4$ , and*

$$W_{A \oplus 0}(B \oplus 0) = \bigcup_{\hat{A} \in U(A), 0 \leq \epsilon \leq 1} W_B(\hat{A}(\epsilon)) = \bigcup_{\hat{B} \in U(B), 0 \leq \epsilon \leq 1} W_A(\hat{B}(\epsilon)).$$

*Proof.* It follows from the singular value decomposition of the leading  $2 \times 2$  principal submatrix of  $U$ . ■

**Lemma 4** *Let  $A, B \in M_2$ . If  $W_{A \oplus 0_{n-2}}(B \oplus 0_{n-2}) = W_A(B)$  then  $0 \in W_A(B)$ . Consequently if  $W_{A \oplus 0_{n-2}}(B \oplus 0_{n-2}) = W_A(B)$  for some  $n$ , then it is true for all  $n$ .*

*Proof.* Without loss of generality, we assume  $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{11} \end{pmatrix}$  where  $a_{11} = \frac{1}{2} \text{tr } A$ . By Lemma 3, we have

$$\begin{aligned} & \left( \frac{1 + \epsilon^2}{2} \right) a_{11} \text{tr } B + W_{B - (\frac{1}{2} \text{tr } B)I}(A(\epsilon)) = W_B(A(\epsilon)) \\ & \subseteq W_B(A) = a_{11} \text{tr } B + W_{B - (\frac{1}{2} \text{tr } B)I}(A) \end{aligned}$$

hence

$$\left( \frac{-1 + \epsilon^2}{2} \right) a_{11} \text{tr } B + W_{B - (\frac{1}{2} \text{tr } B)I}(A(\epsilon)) \subseteq W_{B - (\frac{1}{2} \text{tr } B)I}(A). \quad (1)$$

Note that

$$W_{B - (\frac{1}{2} \text{tr } B)I}(A) = W_{B - (\frac{1}{2} \text{tr } B)I} \left( \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} \right).$$

Note also that  $A(\epsilon) = \begin{pmatrix} a_{11} & \epsilon a_{12} \\ \epsilon a_{21} & \epsilon^2 a_{11} \end{pmatrix}$  is unitarily similar to  $\begin{pmatrix} \epsilon^2 a_{11} & \epsilon a_{12} \\ \epsilon a_{21} & a_{11} \end{pmatrix}$ , and so

$$\begin{aligned} \epsilon W_{B - (\frac{1}{2} \text{tr } B)I}(A) &= W_{B - (\frac{1}{2} \text{tr } B)I} \left( \begin{pmatrix} 0 & \epsilon a_{12} \\ \epsilon a_{21} & 0 \end{pmatrix} \right) \\ &= W_{B - (\frac{1}{2} \text{tr } B)I} \left( \begin{pmatrix} \frac{1}{2}(1 + \epsilon^2)a_{11} & \epsilon a_{12} \\ \epsilon a_{21} & \frac{1}{2}(1 + \epsilon^2)a_{11} \end{pmatrix} \right) \\ &\subseteq W_{B - (\frac{1}{2} \text{tr } B)I}(A(\epsilon)). \end{aligned}$$

Therefore (1) is possible only if

$$\left( \frac{-1 + \epsilon^2}{2} \right) a_{11} \text{tr } B \in (1 - \epsilon) W_{B - (\frac{1}{2} \text{tr } B)I}(A)$$

which implies

$$\left( \frac{-1 - \epsilon}{2} \right) a_{11} \text{tr } B \in W_{B - (\frac{1}{2} \text{tr } B)I}(A) = W_B(A) - a_{11} \text{tr } B.$$

Setting  $\epsilon = 1$ , we have  $0 \in W_B(A)$ . ■

**Lemma 5** *Let  $A, B \in M_2$ . If  $W_{A \oplus 0_{n-2}}(B \oplus 0_{n-2}) = W_A(B)$  then the largest possible  $\alpha$  such that  $\alpha W(A)W(B) \subseteq W_A(B)$  satisfies  $1 \leq \alpha \leq 4$ .*

*Proof.* Let  $a \in W(A)$ , then there exists  $\hat{A} = \begin{pmatrix} a & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in U(A)$ . By Lemma 3, we have  $aW(B) = W_{\hat{A}(0)}(B) \subseteq W_A(B)$ .

By Lemma 1, we have  $W_A(B) = 2W(\hat{A} \circ \hat{B})$  for some  $\hat{A} \in U(A)$  and  $\hat{B} \in U(B)$ . Therefore, if  $\alpha W(A)W(B) \subseteq W_A(B)$ , we have  $\alpha r(A)r(B) \leq 2r(\hat{A} \circ \hat{B})$ . However,  $r(\hat{A} \circ \hat{B}) \leq 2r(\hat{A})r(\hat{B})$  [5, Corollary 1.7.25]. Therefore,  $\alpha r(A)r(B) \leq 4r(A)r(B)$  and thus  $\alpha \leq 4$ .  $\blacksquare$

The lower bound and the upper bound for  $\alpha$  are both sharp. The lower bound is sharp because  $W_{E_{11}}(E_{11}) = [0, 1] = W(E_{11})W(E_{11})$  and the upper bound is sharp because  $W_{E_{12}}(E_{12}) = 4W(E_{12})W(E_{12})$ .

**Lemma 6** *Let  $A, B \in M_2$ . If  $W_{A \oplus 0_{n-2}}(B \oplus 0_{n-2}) = W_A(B)$  then*

$$W_B(A_0) = W_{B \oplus 0_{n-2}}(A_0 \oplus 0_{n-2}) = W_A(B_0) = W_{A_0}(B_0 \oplus 0_{n-2}).$$

*Proof.* By Lemma 3, we have  $W_{\hat{A}(\epsilon)}(B) \subseteq W_A(B)$  for any  $\hat{A} \in U(A)$ , which implies, by Lemma 1, that

$$W_{\hat{A}(\epsilon)}\left(B - \left(\frac{1}{2}\text{tr } B\right)I\right) \subseteq W_A\left(B - \left(\frac{1}{2}\text{tr } B\right)I\right)$$

for any  $\hat{A} \in U(A)$ , and then by Lemma 3 again, we have

$$W_A\left(B - \left(\frac{1}{2}\text{tr } B\right)I\right) = W_{A \oplus 0_{n-2}}\left(\left(B - \left(\frac{1}{2}\text{tr } B\right)I\right) \oplus 0_{n-2}\right).$$

$\blacksquare$

### 3 Main Results

An implication of Lemma 4 is that we only need to consider the case  $n = 3$ .

First of all, we have a sufficient condition.

**Theorem 7** *Let  $A, B \in M_2$ . Suppose*

$$(\text{tr } A)W(B) + (\text{tr } B)W(A) - (\text{tr } A)(\text{tr } B) \subseteq W_A(B),$$

*or equivalently*

$$(\text{tr } A)W(B_0) + (\text{tr } B)W(A_0) - \frac{1}{2}(\text{tr } A)(\text{tr } B) \subseteq W_{A_0}(B_0)$$

*then  $W_A(B) = W_{A \oplus 0}(B \oplus 0)$ .*

*Proof.* Let  $\hat{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & \mathbf{tr} A - a_{11} \end{pmatrix} \in U(A)$  and  $\hat{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & \mathbf{tr} B - b_{11} \end{pmatrix} \in U(B)$ .

We have

$$\begin{aligned}
& \mathbf{tr}(\hat{A}\hat{B}(\epsilon)) \\
&= a_{11}b_{11} + \epsilon(a_{12}b_{21} + a_{21}b_{12}) + \epsilon^2(\mathbf{tr} A - a_{11})(\mathbf{tr} B - b_{11}) \\
&= \frac{1}{2}(1 + \epsilon^2) \left( a_{11}b_{11} + \frac{2\epsilon}{1 + \epsilon^2}(a_{12}b_{21} + a_{21}b_{12}) + (\mathbf{tr} A - a_{11})(\mathbf{tr} B - b_{11}) \right) \\
&\quad + \frac{1}{2}(1 - \epsilon^2)(a_{11}\mathbf{tr} B + b_{11}\mathbf{tr} A - (\mathbf{tr} A)(\mathbf{tr} B)) \\
&= \frac{1}{2}(1 + \epsilon^2) \mathbf{tr} \left( \hat{A} \begin{pmatrix} b_{11} & \frac{2\epsilon}{1 + \epsilon^2}b_{12} \\ \frac{2\epsilon}{1 + \epsilon^2}b_{21} & \mathbf{tr} B - b_{11} \end{pmatrix} \right) \\
&\quad + \frac{1}{2}(1 - \epsilon^2)(a_{11}\mathbf{tr} B + b_{11}\mathbf{tr} A - (\mathbf{tr} A)(\mathbf{tr} B)) \\
&\in W_A(B) \quad \text{by Lemma 2 and the assumption.}
\end{aligned}$$

Hence by Lemma 3, we have  $W_A(B) = W_{A \oplus 0}(B \oplus 0)$ . ■

Let's replace the condition in Theorem 7 with a stronger one to make it easier to apply.

**Theorem 8** *Let  $A, B \in M_2$ . Suppose the disc centered at 0 with radius  $|\mathbf{tr} A|r(B_0) + |\mathbf{tr} B|r(A_0) + \frac{1}{2}|\mathbf{tr} A||\mathbf{tr} B|$  lies inside  $W_{A_0}(B_0)$ , then  $W_A(B) = W_{A \oplus 0}(B \oplus 0)$ .*

*Proof.* It is a direct consequence of Theorem 7. ■

It turns out that the sufficient condition is also necessary if one of the two matrices is trace 0.

**Theorem 9** *Let  $A, B \in M_2$ . Suppose  $\mathbf{tr} B = 0$ . Then  $W_A(B) = W_{A \oplus 0}(B \oplus 0)$  iff  $(\mathbf{tr} A)r(B) \subseteq W_A(B)$ .*

*Proof.* Sufficiency follows from Theorem 7.

Suppose that  $W_A(B) = W_{A \oplus 0}(B \oplus 0)$ . If there exists  $\beta \in W(B)$  such that  $(\mathbf{tr} A)\beta \notin W_A(B)$ , then there exists  $\theta$  such that  $\mathbf{Re} e^{i\theta} \mathbf{tr}(A)\beta > \mathbf{Re} e^{i\theta} x$  for all  $x \in W_A(B)$ . Without loss of generality, we can assume that  $\hat{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$  and  $\hat{B} = \begin{pmatrix} \beta & b_{12} \\ b_{21} & -\beta \end{pmatrix}$  are such that  $\mathbf{Re} e^{i\theta} \mathbf{tr}(\hat{A}\hat{B})$  is the largest possible in  $W_A(B)$ .

As  $W_A(B) = W_{A \oplus 0}(B \oplus 0)$ , we know that  $\mathbf{tr}(\hat{A}\hat{B}(\epsilon)) \in W_A(B)$  and hence

$$\mathbf{Re} e^{i\theta}(a_{11}\beta + \epsilon(a_{12}b_{21} + a_{21}b_{12}) - \epsilon^2 a_{22}\beta) \leq \mathbf{Re} e^{i\theta}(a_{11}\beta + (a_{12}b_{21} + a_{21}b_{12}) - a_{22}\beta).$$

Reorganizing, we have

$$\mathbf{Re} e^{i\theta}((1 - \epsilon)(a_{12}b_{21} + a_{21}b_{12}) - (1 - \epsilon^2)a_{22}\beta) \geq 0$$

and so

$$\mathbf{Re} e^{i\theta}((a_{12}b_{21} + a_{21}b_{12}) - (1 + \epsilon)a_{22}\beta) \geq 0.$$

Thus, put  $\epsilon = 1$ , we have

$$\mathbf{Re} e^{i\theta}((a_{12}b_{21} + a_{21}b_{12}) - a_{22}\beta) \geq \mathbf{Re} e^{i\theta}a_{22}\beta.$$

Therefore

$$\mathbf{Re} e^{i\theta} \mathbf{tr}(\hat{A}\hat{B}) \geq \mathbf{Re} e^{i\theta}(a_{11}\beta + a_{22}\beta) = \mathbf{Re} e^{i\theta}(\mathbf{tr} A)\beta$$

which is a contradiction. ■

As a corollary, we have a necessary condition.

**Theorem 10** *Let  $A, B \in M_2$ . If  $W_A(B) = W_{A \oplus 0}(B \oplus 0)$  then*

$$(\mathbf{tr} A)W(B) \cup (\mathbf{tr} B)W(A) \subseteq W_A(B).$$

*Proof.* By Lemma 6, we know

$$W_A \left( B - \left( \frac{1}{2} \mathbf{tr} B \right) I \right) = W_{A \oplus 0_{n-2}} \left( \left( B - \left( \frac{1}{2} \mathbf{tr} B \right) I \right) \oplus 0_{n-2} \right),$$

and then by Theorem 9, we have

$$\begin{aligned} & (\mathbf{tr} A)W(B) - \frac{1}{2}(\mathbf{tr} A)(\mathbf{tr} B) = (\mathbf{tr} A)W \left( B - \left( \frac{1}{2} \mathbf{tr} B \right) I \right) \\ & \subseteq W_A \left( B - \left( \frac{1}{2} \mathbf{tr} B \right) I \right) = W_A(B) - \frac{1}{2}(\mathbf{tr} A)(\mathbf{tr} B). \end{aligned}$$

Likewise we have  $(\mathbf{tr} B)W(A) \subseteq W_A(B)$ . ■

If one of the matrices is Hermitian, then we also have a necessary and sufficient condition.

**Theorem 11** *Let  $A, B \in M_2$ . Suppose  $B$  is Hermitian. Then  $W_A(B) = W_{A \oplus 0}(B \oplus 0)$  is equivalent to  $(\mathbf{tr} A)W(B) \cup (\mathbf{tr} B)W(A) \subseteq W_A(B)$ . If, in addition, both  $A$  and  $B$  are nonzero matrices, then it is also equivalent to  $0 \in W(A) \cap W(B)$ .*

*Proof.* It suffices to consider  $A, B \neq 0$ . Let  $b_1 \geq b_2$  be the two eigenvalues of  $B$  and then  $W_A(B) = (b_2 - b_1)W(A) + b_1 \mathbf{tr} A = (b_1 - b_2)W(A) + b_2 \mathbf{tr} A$ . By Theorem 10,  $W_A(B) = W_{A \oplus 0}(B \oplus 0)$  implies  $(\mathbf{tr} A)W(B) \cup (\mathbf{tr} B)W(A) \subseteq W_A(B)$ .

Suppose  $(\mathbf{tr} A)W(B) \cup (\mathbf{tr} B)W(A) \subseteq W_A(B)$ . Therefore  $(b_1 + b_2)W(A) = (\mathbf{tr} B)W(A) \subseteq W_A(B)$  which, by considering the area of the two sets, implies that  $|b_1 + b_2| \leq b_1 - b_2$  and hence  $b_1 \geq 0 \geq b_2$ .  $b_1 \mathbf{tr} A \in W_A(B) = (b_2 - b_1)W(A) + b_1 \mathbf{tr} A$  implies that  $0 \in W(A)$ .

$0 \in W(A) \cap W(B)$  implies that  $b_1 \geq 0 \geq b_2$  and  $0 \in W(0.5(e^{i\theta}A + e^{-i\theta}A^*))$  and consequently  $W_A(B) = W_{A \oplus 0}(B \oplus 0)$ . ■

## 4 New Convex Pairs

**Corollary 12** *Let  $A, B \in M_2$  such that  $\mathbf{tr} A = \mathbf{tr} B = 0$ . We have  $W_A(B) = W_{A \oplus 0}(B \oplus 0)$ . Moreover, if both  $A$  and  $B$  are nonzero, then the largest possible  $\alpha$  such that  $\alpha W(A)W(B) \subseteq W_A(B)$  satisfies  $2 \leq \alpha \leq 4$ .*

*Proof.* That  $W_A(B) = W_{A \oplus 0}(B \oplus 0)$  follows from Theorem 9.

Let  $a \in W(A)$  and  $b \in W(B)$ , then there exists  $\hat{A} = \begin{pmatrix} a & a_{12} \\ a_{21} & -a \end{pmatrix} \in U(A)$  and  $\hat{B} = \begin{pmatrix} b & b_{12} \\ b_{21} & -b \end{pmatrix} \in U(B)$ . Hence by Lemma 2, we have

$$2ab = \mathbf{tr} \left( \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix} \right) \in W_A(B).$$

Hence we have  $2W(A)W(B) \subseteq W_A(B)$ .

The upper bound follows from Lemma 5. ■

The lower bound and the upper bound for  $\alpha$  are both sharp. The lower bound is sharp since  $W_{E_{11}-E_{22}}(E_{11} - E_{22}) = 2W(E_{11} - E_{22})W(E_{11} - E_{22})$ . The upper bound is sharp, the example is the same as that of Lemma 5.

**Example 1**  $W \left( \begin{pmatrix} 2+i & 3 & 0 \\ 1-2i & -2-i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1+i & 2-i & 0 \\ 1-2i & -1-i & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$  is convex.

**Corollary 13** *Let  $A, B \in M_2$ . If  $\mathbf{tr} A = 0$  and  $\frac{1}{2}\mathbf{tr} B \in W(B_0)$  then  $W_A(B) = W_{A \oplus 0}(B \oplus 0)$ .*

*Proof.* It follows Theorem 9 and that  $2W(A)W(B_0) \subseteq W_A(B_0)$ . ■



**Example 2**  $W\left(\begin{pmatrix} 2+i & 3 & 0 \\ 1-2i & -2-i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2-i & 0 \\ 1-2i & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$  is convex.

**Corollary 14** If  $A = B = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$  with  $|a| \leq \frac{\sqrt{3}-1}{2}|b|$  then  $W_A(B) = W_{A \oplus 0}(B \oplus 0)$  is a circular disc centered at  $2a^2$  of radius  $|b|^2$ .

*Proof.* Without loss of generality, we let  $0 \leq a \leq \frac{\sqrt{3}-1}{2}$  and  $b = 1$ . Note that  $W(A_0) = W(B_0)$  is a circular disc centered at 0 of radius 0.5 and  $W_{A_0}(B_0)$  is a unit disc centered at 0.

For  $v \in (\mathbf{tr} A)W(B_0) + (\mathbf{tr} B)W(A_0) - \frac{1}{2}(\mathbf{tr} A)(\mathbf{tr} B) = 2a(W(B_0) + W(A_0) - a)$ , we have

$$|v| \leq 2|a|(2r(A_0) + |a|) = 2|a|(1 + |a|) \leq 1,$$

thus  $v \in W_{A_0}(B_0)$ .

By Theorem 8,  $W_A(B) = W_{A \oplus 0}(B \oplus 0)$ . ■

**Example 3** If  $C = \begin{pmatrix} 0.7 & 2 & 0 \\ 0 & 0.7 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  then  $W(C, C^*)$  is a circular disc centered at 0.98. This answers Problem 3.

**Corollary 15** If  $A = B = \begin{pmatrix} a & b \\ 1 & a \end{pmatrix}$  with  $b > 1$  and  $2|a|^2 + (1+b)|a| - (b^2 - 1) \leq 0$  then  $W_A(B) = W_{A \oplus 0}(B \oplus 0)$

*Proof.* Note that  $r(A_0) = r(B_0) = \frac{1+b}{2}$ . By Lemma 1, we know that for the circular disc centered at 0 with radius  $b^2 - 1$  is contained in  $W_{A_0}(B_0)$ .

For  $v \in (\mathbf{tr} A)W(B_0) + (\mathbf{tr} B)W(A_0) - \frac{1}{2}(\mathbf{tr} A)(\mathbf{tr} B) = 2a(W(B_0) + W(A_0) - a)$ , we have

$$|v| \leq 2|a|(2r(A_0) + |a|) = |a|(1 + b + 2|a|) \leq b^2 - 1,$$

thus  $v \in W_{A_0}(B_0)$ .

By Theorem 8,  $W_A(B) = W_{A \oplus 0}(B \oplus 0)$ . ■

**Example 4** If  $C = \begin{pmatrix} i & 3 & 0 \\ 1 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}$  then  $W(C, C)$  is convex. Note that the eigenvalues of  $C$  are not collinear, this answers Problem 2.

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